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Noncommutative Bayesian Statistical Inference From a Wedge of a Bifurcate Killing Horizon

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Expanding the remark 5.2.7 of Segre (Segre, G. (2002). *Algorithmic Information Theoretic Issues in Quantum Mechanics*, PhD Thesis, Dipartimento di Fisica Nucleare e Teorica, Pavia, Italy. quant-ph/0110018.) the noncommutative bayesian statistical inference from one wedge of a bifurcate Killing horizon is analyzed looking at its interrelation with the Unruh effect.

KEY WORDS: noncommutative statistical inference; Unruh effect.

NOTATION

$x = y$	x is equal to y
$x := y$	x is defined as y
card(S)	cardinality of S
f_*	differential map of f
f^*	pull-back of f
$\mathcal{L}_X$	Lie derivative w.r.t. X
Is$[(M, g_{ab})]$	isometry group of the space-time (M, g_{ab})
$\Gamma(T^{(r,s)}M)$	sections of the (r, s)-tensor bundle over M
$I^-(S)$	chronological past of the space-time's region S
$I^+(S)$	chronological feature of the space-time's region S
$J^-(S)$	causal past of the space-time's region S
$J^+(S)$	causal future of the space-time's region S
$D^-(S)$	past domain of dependence of the space-time's region S
$D^+(S)$	future domain of dependence of the space-time's region S
$D(S)$	domain of dependence of the space-time's region S
$S(A)$	space of the states over a W^*-algebra
τ_{unbiased}	unbiased state
$A^W_{(M,g_{ab})}$	Weyl algebra of the space-time (M, g_{ab})
$S_H(A^W_{(M,g_{ab})})$	Hadamard states over $A^W_{(M,g_{ab})}$

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σ_t^{ω}	modular group of the state ω
AUT(A)	automorphisms of the W^*-algebra A
INN(A)	inner automorphisms of the W^*-algebra A
OUT(A)	outer automorphisms of the W^*-algebra A
GR – AUT(G, A)	automorphisms' group of A representing G
GR – INN(G, A)	inner automorphisms' groups of A representing G
GR – OUT(G, A)	outer automorphisms' group of A representing G
Ⓢ	semidirect product of groups

1. NONCOMMUTATIVE BAYESIAN STATISTICAL INFERENCE

Noncommutative Bayesian Statistical Inference as introduced by Miklos Redei in the 8th chapter "*Quantum conditional and quantum conditional probability*" of Redei (1998), is based on the following analysis.

Given a classical probability space (X, σ, μ) let us suppose to be a statistician having access only to the partial information concerning the probability of an event $B \in \sigma$ and whose goal is to estimate the unknown probability $\mu(A)$ of an arbitrary event $A \in \sigma$.

The Bayesian recipe prescribes that, before using even the partial information he has, the more natural estimation of $\mu(A)$ is the one introducing no bias, i.e.

$$\mu_{\text{A PRIORI}}(A) := P_{\text{unbiased}}(A) \tag{1.1}$$

i.e. the normalized distribution over (X, σ) whether card$(X) < \aleph_0$ or the Lebesgue measure over (X, σ) whether card$(X) = \aleph_1$.

Let us observe that in the case card$(X) = \aleph_0$ the unbiased probability measure over (X, σ) doesn't exist so that the Bayesian strategy of statistical inference is not defined in that case.

The acquisition of the partial information he can access results, according to the Bayesian recipe, in the following ansatz:

$$\mu_{\text{A PRIORI}}(A) := P_{\text{unbiased}}(A) \rightarrow \mu_{\text{A POSTERIORI}}(A) := \frac{\mu_{\text{A PRIORI}}(A \cap B)}{\mu(B)} \tag{1.2}$$

Let us now recall the Basic Theorem of Noncommutative Probability stating that the category having as objects the classical probability spaces and as morphisms their automorphisms is equivalent to the category having as objects the algebraic commutative probability spaces and as morphisms their automorphisms.

Such a theorem naturally leads to a noncommutative generalization of the Bayesian recipe consisting in

1. the recasting of Eq. (1.2) in the language of algebraic probability spaces;
2. the generalization to noncommutative probability spaces.

Given the algebraic commutative probability space (A, ω) with

$$A := L^{\infty}(X, \sigma, \mu) \tag{1.3}$$

$$\omega(a) := \int_X d\mu\, a \quad a \in A \tag{1.4}$$

let us suppose that the statistician has access only to the information concerning a sub-σ-algebra $\sigma_{\text{accessible}}$ of σ.

Introduced the W^* – algebra:

$$A_{\text{accessible}} := L^{\infty}(X, \sigma_{\text{accessible}}, \mu_{\text{accessible}}) \tag{1.5}$$

where

$$\mu_{\text{accessible}} := \mu|_{\sigma_{\text{accessible}}} \tag{1.6}$$

and the associated state $\omega_{\text{accessible}} \in S(A_{\text{accessible}})$:

$$\omega_{\text{accessible}}(a) := \int_X d\mu_{\text{accessible}}\, a \quad a \in A_{\text{accessible}} \tag{1.7}$$

we can express the Bayesian recipe in the following way:

1. Before using even the partial information that is accessible to him, the better *a priori estimation* of ω the statistician can perform consists in introducing no bias, assuming that

$$\omega_{\text{A PRIORI}} := \tau_{\text{unbiased}}$$

where τ_{unbiased} is the unbiased state over A:

$$\tau_{\text{unbiased}}(a) := \int_X dP_{\text{unbiased}}\, a \quad a \in A \tag{1.8}$$

2. The adoption of the available information may be encoded in the passage from the *a priori estimation* to the *a posteriori estimation* of ω specified by the *Bayes rule*:

$$\omega_{\text{A PRIORI}}(\cdot) = \tau_{\text{unbiased}}(\cdot) \to \omega_{\text{A POSTERIORI}}(\cdot) := \omega_{\text{accessible}}(E_{\text{A PRIORI}} \cdot\,) \tag{1.9}$$

where $E_{\text{A PRIORI}} : A \to A_{\text{accessible}}$ is the *conditional expectation w.r.t.* $A_{\text{accessible}}\omega_{\text{A PRIORI}}$*-invariant.*

The Basic Theorem of Noncommutative Probability allows to generalize immediately such a recipe to the noncommutative case in which a statistician has access only to the noncommutative probability subspace $(A_{\text{accessible}}, \omega_{\text{accessible}})$ of a larger noncommutative probability space (A, ω):

$$\omega_{\text{accessible}} := \omega|_{A_{\text{accessible}}} \tag{1.10}$$

resulting in the following *noncommutative bayesian recipe*:

1. Before using even the partial information that is accessible to him, the better *a priori estimation* of ω consists in introducing no bias, assuming that

$$\omega_{\text{A PRIORI}} := \tau_{\text{unbiased}}$$

where τ_{unbiased} is the noncommutative unbiased probability distribution over A, namely the tracial state on it.

2. The adoption of the available information may be encoded in the passage from the *a priori estimation* to the *a posteriori estimation* of ω specified by the *noncommutative Bayes rule*:

$$\omega_{\text{A PRIORI}}(\cdot) = \tau_{\text{unbiased}}(\cdot) \to \omega_{\text{A POSTERIORI}}(\cdot) := \omega_{\text{accessible}}(E_{\text{A PRIORI}}\cdot) \tag{1.11}$$

where $E_{\text{A PRIORI}} : A \to A_{\text{accessible}}$ is the *conditional expectation* w.r.t. $A_{\text{accessible}}$ $\omega_{\text{A PRIORI}}$-*invariant.*

Let us now observe that to the feasibility condition for such a statistical inference already present in the commutative case and requiring the existence of the unbiased probability distribution, another constraint has to be added in the noncommutative case: according to Takesaki Theorem $E_{\text{A PRIORI}}$ exists if and only if the following *modular constraint* is satisfied:

$$\sigma_t^{\omega_{\text{A PRIORI}}}(a) \in A_{\text{accessible}} \; \forall a \in A_{\text{accessible}}, \quad \forall t \in \mathbb{R} \tag{1.12}$$

where $\sigma_t^{\omega_{\text{A PRIORI}}}$ denotes the modular group of $\omega_{\text{A PRIORI}}$.

Let us observe, with this regard, that since the modular constraint is not satisfied for any sub-W^*-algebra, the philosophical subjectivistic viewpoint consistent in the commutative case, cannot be generalized to the noncommutative case.

2. NONCOMMUTATIVE STATISTICAL INFERENCE FROM A WEDGE OF A BIFURCATE KILLING HORIZON

Given a space-time, i.e., a four-dimensional lorentzian manifold (M, g_{ab})[2] (Kobayashi and Nomizu, 1996; Wald, 1984, 1994), let us suppose that it admits a bifurcate Killing horizon, i.e. a bidimensional space-like surface S such that there exist a Killing vector field X^a vanishing on it:

$$\mathcal{L}_{X^a} g_{ab} = 0 \tag{2.1}$$

$$X^a(p) = 0 \quad \forall p \in S \tag{2.2}$$

[2] I will follow Penrose abstract index notation as explained, e.g., in Wald (1984).

Let us suppose, furthermore, that S is a Cauchy surface of (M, g_{ab}),[3] i.e. that its domain of dependence is the whole M:

$$D(S) = M \tag{2.3}$$

Denoted by h_A and h_B the two null surfaces generated by the null geodesics orthogonal to S, M may be expressed as the union of four disjoint wedges:

$$M = \cup_{i=1}^{4} W_i \tag{2.4}$$

$$W_1 := I^{-}(h_A) \cap I^{+}(h_B) \tag{2.5}$$

$$W_2 := I^{+}(h_A) \cap I^{-}(h_B) \tag{2.6}$$

$$W_3 := J^{+}(S) \tag{2.7}$$

$$W_4 := J^{-}(S) \tag{2.8}$$

Let us now suppose to be a statistician living in a Universe whose Physics sufficiently far from Planck's scale is described by a quantum field theory on (M, g_{ab}), specified by the set of local observables' algebras $\{A_O\}_{O \subseteq M}$ obeying Dimock's axioms (i.e., Dimock's generalization to curved space-time of Haag–Kastler's axioms) (Dimock, 1980; Haag, 1996; Verch, 2002) whose world-line is a flow line of the above Killing vector field X^a.

Supposing he can access only the state of affairs concerning the physical observables localized in W_1 his objective is to make a statistical inference concerning the state of affairs outside W_1.

Denoting by

$$A_{\text{accessible}} := A_{W_1} \tag{2.9}$$

the algebra of observables that is accessible to him, his objective is to estimate the true state $\omega \in S(A^W_{(M,g_{ab})})$ of the noncommutative probability space $(A^W_{(M,g_{ab})}, \omega)$ describing the Universe in the assumed classical-background approximation, $A^W_{(M,g_{ab})}$ denoting the Weyl algebra of (M, g_{ab}), from the knowledge of the information accessible to him, codified by the *accessible state* defined as the restriction of ω to the *accessible algebra*:

$$\omega_{\text{accessible}} := \omega|_{A_{\text{accesible}}} \in S(A_{\text{accesible}}) \tag{2.10}$$

Noncommutative Bayesian Statistical Theory, as described in the previous section, would prescribe to him to adopt the following recipe:

[3] We have implicitly assumed that (M, g_{ab}) is globally hyperbolic and, hence, admits Cauchy surfaces. While in Classical General Relativity the status of the Strong Cosmic Censorship Conjecture stating that any "physical" space-time is globally hyperbolic is dubious, it is strongly dubious whether a Quantum Field Theory on a nonglobally hyperbolic space-time may be consistently formalized.

1. Before using even the partial information that is accessible to him, the better *a priori estimation* of ω consists in introducing no bias, assuming that

$$\omega_{\text{A PRIORI}} := \tau_{\text{unbiased}}$$

where τ_{unbiased} is the noncommutative unbiased probability distribution over $A^{W}_{(M,g_{ab})}$, namely the tracial state on it.
2. The adoption of the available information may be encoded in the passage from the *a-priori estimation* to the *a posteriori estimation* of ω specified by the *noncommutative Bayes rule*:

$$\omega_{\text{A PRIORI}}(\cdot) = \tau_{\text{unbiased}}(\cdot) \rightarrow \omega_{\text{A POSTERIORI}}(\cdot) := \omega_{\text{accessible}}(E_{\text{A PRIORI}}\cdot) \tag{2.11}$$

where $E_{\text{A PRIORI}} : A^{W}_{(M,g_{ab})} \mapsto A_{\text{accessible}}$ is the *conditional expectation w.r.t.* $A_{\text{accessible}}$ $\omega_{\text{A PRIORI}}$*-invariant.*

Let us observe, first of all, that, according to Takesaki Theorem, the existence of the involved conditional expectation and, hence, the feasibility of the Bayesian statistical inference, requires the assumption of the following *modular constraint*:

$$\sigma_{t}^{\omega_{\text{A PRIORI}}}(a) \in A_{\text{accessible}} \quad \forall a \in A_{\text{accessible}}, \quad \forall t \in \mathbb{R} \tag{2.12}$$

Let us observe, furthermore, that since the Weyl's algebra $A^{W}_{(M,g_{ab})}$ of (M, g_{ab}) is generally not finite, the unbiased noncommutative probability measure τ_{unbiased} doesn't exist.

It must be observed, at this point, that there exists, anyway, an a priori information that the statistician can adopt: the fact that the expectation value $\langle T_{ab}\rangle$ of the stress–energy operator T_{ab} must be well-defined in order of making the back-reaction's semiclassical Einstein equation:

$$G_{ab} = 8\pi \langle T_{ab}\rangle \tag{2.13}$$

well-defined too, resulting in the condition that ω is an *Hadamard state* over the Weyl's algebra $A^{W}_{(M,g_{ab})}$ of (M, g_{ab}):

$$\omega \in S_{H}\left(A^{W}_{(M,g_{ab})}\right) \tag{2.14}$$

Consequentially it is natural for the statistician to assume that $\omega_{\text{A PRIORI}}$ is an Hadamard state too.

$$\omega_{\text{A PRIORI}} \in S_{H}\left(A^{W}_{(M,g_{ab})}\right) \tag{2.15}$$

Furthermore he a priori knows that some information about the observables not accessible to him may be recovered by the information concerning $A_{\text{accessible}}$

through the condition:

$$\exists\{\alpha_g\} \in \mathrm{GR} - \mathrm{INN}\left[\mathrm{Is}(M, g_{ab}), A^W_{(M,g_{ab})}\right] : \alpha_g(A_O) = A_{gO} \quad \forall O \subset W_1 \tag{2.16}$$

Consequentially it is natural, for the statistician, to choose the a priori state as much $\mathrm{Is}[(M, g_{ab})]$-invariant as possible.

To understand how this two constraints concretely work as to the determination of $\omega_{\text{A PRIORI}}$ it is useful to start analyzing the simpler cases.

3. THE MINKOWSKI CASE

Let us start analyzing the simplest particular case in which (M, g_{ab}) is the Minkowski space-time:

$$M := \mathbb{R}^4 \tag{3.1}$$

$$g_{ab} := \eta_{ab} := \eta_{\mu\nu} dx^\mu \otimes dx^\nu \tag{3.2}$$

$$\eta_{\mu\nu} := \mathrm{diag}(-1, 1, 1, 1) \tag{3.3}$$

The isometries-group of Minkowski space-time is the Poincaré group $SO(1, 3)$ Ⓢ$\mathbb{R}^4$ generated by the 10 Killing vector fields:

$$T^\mu_{(i)} := \delta^\mu_i \quad i = O, \ldots, 3 \tag{3.4}$$

$$L_{\mu\nu} := x_\mu \partial_\nu - x_\nu \partial_\mu \quad \nu > \mu = 0, \ldots, 3 \tag{3.5}$$

Let us then observe that the surface

$$S := \{x^\mu \in \mathbb{R}^4 : x^0 = x^1 = 0\} \tag{3.6}$$

is a bifurcate Killing horizon for the Killing vector field:

$$X^a := L_{01} \tag{3.7}$$

generating boosts in the direction x^1. Let us denote by α_t the inner automorphisms' group representing the one-dimensional subgroup i_t of $\mathrm{Is}[(\mathbb{R}^4, \eta_{ab})]$ generated by X^a.

Since the domain of dependence $D(S)$ of the surface S is such that

$$D(S) = \mathbb{R}^4 \tag{3.8}$$

S is a Cauchy surface, so that, according to the general analysis previously introduced, one has the splitting of the Minkowski space-time in the four wedges specified by Eq. (2.4) with

$$h_A = \{x^\mu \in \mathbb{R}^4 : x^0 = x^1\} \tag{3.9}$$

$$h_B = \{x^\mu \in \mathbb{R}^4 : x^0 = -x^1\} \tag{3.10}$$

$$W_1 = \{x^\mu \in \mathbb{R}^4 : |x^1| < x^0, x^0 > 0\} \tag{3.11}$$

$$W_2 = \{x^\mu \in \mathbb{R}^4 : |x^1| < x^0, x^0 < 0\} \tag{3.12}$$

$$W_3 = \{x^\mu \in \mathbb{R}^4 : |x^1| \geq x^0, x^1 > 0\} \tag{3.13}$$

$$W_4 = \{x^\mu \in \mathbb{R}^4 : |x^1| \geq x^0, x^1 < 0\} \tag{3.14}$$

Since X^a is time-like in the two wedges W_1 and W_2 its flow i_t represents possible world-lines of a massive observer such as our statistician; we will suppose, precisely, that the statistician's world-line is an integral curve of X^a contained in W_1.

Following the condition enunciated in the last section, among the possible Hadamard states that our statistician may choose as a-priori state, the more natural one is the restriction to A_{W_1} of the only $\text{Is}(\mathbb{R}^4, \eta_{ab})$-invariant one, i.e., the vacuum state $\omega_{(0)}$:

$$\omega_{\text{A PRIORI}} := \omega_{(0)}|_{A_{W_1}} \in S(A_{W_1}) \tag{3.15}$$

The Unruh effect, consisting in the fact that, in the case $\omega = \omega_{(0)}$ in which the state to estimate is the vacuum one, such a vacuum state appears to the statistician following the flow i_t of L_{01} as a thermal bath, has been explained by Geoffrey Sewell in terms of Modular Theory through the Bisognano–Wichmann theorem (Narnhofer *et al.*, 1998), Haag (1996) stating that $\omega_{\text{A PRIORI}}$ is an α_t-KMS-state at $\beta = 2\pi$.

Let us now recall that the feasibility of the statistical inferential problem is itself ruled by the modular group of $\omega_{\text{A PRIORI}}$ through the *modular constraint* of Eq. (2.12) whose satisfaction, in the present case:

$$\sigma_t^{\omega_0}(a) \in A_{W_1} \quad \forall a \in A_{W_1} \tag{3.16}$$

should follow by the i_t-invariance of ω_0, by the fact that $(W_1, \eta_{ab}|_{\Gamma(T^{(0,2)}W_1)})$ is a globally hyperbolic space-time for its own and by the fact that

$$\alpha_t A_O = A_{i_t O} \quad \forall O \subset W_1 \tag{3.17}$$

Modular Theory tells us, furthermore, that $\omega_{\text{A PRIORI}}$ is a $\sigma_{-t}^{\omega_{\text{A PRIORI}}}$-KMS state at $\beta = 1$.

This double role of the modular group $\sigma_t^{\omega_{\text{A PRIORI}}}$ could suggest an interpretation of the Unruh effect (stating that, in the case $\omega = \omega_{(0)}$ in which the state to estimate is the vacuum one, our accelerated statistician feels a positive temperature) in terms of the Noncommutative Bayesian Statistical Inference he performs about the whole noncommutative probability space $(A^W_{(\mathbb{R}^4,\eta_{ab})}, \omega)$ having access only to the local information of A_{W_1}.

Such a strategy of statistical inference is codified through the following *modified Bayes recipe*:

1. Before making use even of the partial information that is accessible to him, the better *a priori estimation* of ω consists in assuming as a priori

state the restriction of the vacuum state to the accessible algebra:

$$\omega_{\text{A PRIORI}} := \omega_{(0)}|_{A_{W_1}}$$

2. The adoption of the available information may be encoded in the passage from the *a priori estimation* to the *a posteriori estimation* of ω specified by the *noncommutative Bayes rule*:

$$\omega_{\text{A PRIORI}}(\cdot) = \omega_{(0)}|_{A_{W_1}}(\cdot) \to \omega_{\text{A POSTERIORI}}(\cdot) := \omega_{\text{accessible}}(E_{\text{A PRIORI}}\cdot) \tag{3.18}$$

where $E_{\text{A PRIORI}} : A^{W}_{(\mathbb{R}^4,\eta_{ab})} \mapsto A_{W_1}$ is the *conditional expectation w.r.t.* $A_{W_1}\omega_{\text{A PRIORI}}$*-invariant*.

4. THE DE SITTER CASE

Let us then pass to analyze the case in which (M, g_{ab}) is the De Sitter space-time of unit radius (Narnhofer *et al.*, 1998):

$$M := \{x^{\mu} \in \mathbb{R}^5 : \eta_{\mu\nu}x^{\mu}x^{\nu} = 1\} \tag{4.1}$$

$$g_{ab} := i^{*}\eta_{AB} \tag{4.2}$$

$$\eta_{AB} := \eta_{\mu\nu}dx^{\mu} \otimes dx^{\nu} \tag{4.3}$$

$$\eta_{\mu\nu} := \text{diag}(-1, 1, 1, 1, 1) \tag{4.4}$$

namely the hyperboloid of unit radius embedded in the $(1, 4)$-Minkowskian space-time $(\mathbb{R}^5, \eta_{AB})$ endowed with the lorentzian metric induced by the inclusion (identity) embedding $i : M \mapsto \mathbb{R}^5 : i(p) := p \, \forall p \in M$.

The isometries-group of $(\mathbb{R}^5, \eta_{AB})$

$$\text{Is}[(\mathbb{R}^5, \eta_{AB})]| = SO(1, 4) \circledS \mathbb{R}^5 \tag{4.5}$$

is generated by the 15 Killing vector fields:

$$T^{\mu}_{(i)} := \delta^{\mu}_{i} \quad i = 0, \ldots, 4 \tag{4.6}$$

$$L_{\mu\nu} := x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} \quad \nu > \mu = 0, \ldots, 4 \tag{4.7}$$

While

$$i_{*}L_{\mu\nu} \in \Gamma(TM) \quad \nu = \mu = 0, \ldots, 4 \tag{4.8}$$

one has that

$$i_{*}T^{\mu}_{(i)} \notin \Gamma(TM) \quad i = 0, \ldots, 4 \tag{4.9}$$

It follows that the isometry group of the De Sitter space-time (M, g_{ab}):

$$\text{Is}[(M, g_{ab})] = SO(1, 4) \tag{4.10}$$

is generated by the 10 Killing vector fields:

$$i_* L_{\mu\nu} \in \Gamma(TM) \quad \nu > \mu = 0, \ldots, 4 \tag{4.11}$$

Let us then observe that

$$S := \{x^\mu \in \mathbb{R}^5 : x^0 = x^1 = 0\} \tag{4.12}$$

is a bifurcate Killing horizon for the Killing vector field L_{01} of $(\mathbb{R}^5, \eta_{AB})$ generating boosts in the direction x^1.

Since the domain of dependence $D(S)$ of the surface S is such that,

$$D(S) = \mathbb{R}^5 \tag{4.13}$$

that is S is a Cauchy surface of $(\mathbb{R}^5, \eta_{AB})$, according to the general analysis previously introduced one has the splitting of the (1, 4)-Minkowski space-time in four wedges specified by Eq. (2.4) with

$$h_A = \{x^\mu \in \mathbb{R}^5 : x^0 = x^1\} \tag{4.14}$$

$$h_B = \{x^\mu \in \mathbb{R}^5 : x^0 = -x^1\} \tag{4.15}$$

$$W_1 = \{x^\mu \in \mathbb{R}^5 : |x^1| < x^0, x^0 > 0\} \tag{4.16}$$

$$W_2 = \{x^\mu \in \mathbb{R}^5 : |x^1| < x^0, x^0 < 0\} \tag{4.17}$$

$$W_3 = \{x^\mu \in \mathbb{R}^5 : |x^1| \geq x^0, x^1 > 0\} \tag{4.18}$$

$$W_4 = \{x^\mu \in \mathbb{R}^5 : |x^1| \geq x^0, x^1 < 0\} \tag{4.19}$$

It follows that $S|_M$ is a bifurcate Killing horizon for the Killing vector field:

$$X^a := i_* L_{01} \tag{4.20}$$

of (M, g_{ab}). Since $S|_M$ is a Cauchy surface of (M, g_{ab}):

$$D(S|_M) = M \tag{4.21}$$

one has the splitting of (M, g_{ab}) into the four wedges specified by Eq. (2.4) with

$$H_i := W_i|_M \quad i = 1, \ldots, 4 \tag{4.22}$$

Since the Is(M, g_{ab})-invariance selects again a single state, the vacuum state ω_0, among the Hadamard ones, the prescribed conditions for the selection of the a priori state lead us to the *modified Bayes recipe*:

1. Before making use even of the partial information that is accessible to him, the better *a priori estimation* of ω consists in assuming as a priori-state the restriction of the vacuum state to the accessible algebra:

$$\omega_{\text{A PRIORI}} := \omega_{(0)}|_{A_{H_1}}$$

2. The adoption of the available information may be encoded in the passage from the *a priori estimation* to the *a posteriori estimation* of ω specified by the *noncommutative Bayes rule*:

$$\omega_{\text{A PRIORI}}(\cdot) = \omega_{(0)}|_{A_{H_1}}(\cdot) \to \omega_{\text{A POSTERIORI}}(\cdot) := \omega_{\text{accessible}}(E_{\text{A PRIORI}}\cdot) \tag{4.23}$$

where $E_{\text{A PRIORI}} : A^{W}_{(M,g_{ab})} \mapsto A_{H_1}$ is the *conditional expectation w.r.t.* A_{H_1} $\omega_{\text{A PRIORI}}$*-invariant*.

Let us denote by α_t the inner automorphisms' group representing the one-dimensional subgroup i_t of Is[(M, g_{ab})] generated by X^a.

The Unruh effect, consisting in the fact that, in the case $\omega = \omega_{(0)}$ in which the state to estimate is the vacuum one, such a vacuum state appears to the statistician following the flow of X^a as a thermal bath, has been recasted by Figari, Höegh-Krohn, and Nappi and later by Bros and Moschella in terms of Modular Theory through the Bisognano–Wichmann theorem (Narnhofer *et al.*, 1998), Haag (1996) stating that $\omega_{\text{A PRIORI}}$ is an α_t-KMS-state at $\beta = 2\pi$.

Let us now recall that, exactly as in the Minkowskian case, the feasibility of the statistical inferential problem is itself ruled by the modular group of $\omega_{\text{A PRIORI}}$ through the *modular constraint* of Eq. (2.12) whose satisfaction, in the present case

$$\sigma_t^{\omega_0}(a) \in A_{H_1} \quad \forall a \in A_{H_1} \tag{4.24}$$

should follows by the i_t-invariance of ω_0, by the fact that $(H_1, g_{ab}|_{\Gamma(T^{(0,2)}H_1)})$ is a globally hyperbolic space-time for its own, and by the fact that

$$\alpha_t A_O = A_{i_t O} \quad \forall O \subset H_1 \tag{4.25}$$

Modular Theory, furthermore, tells us that $\omega_{\text{A PRIORI}}$ is a $\sigma_{-t}^{\omega_{\text{A PRIORI}}}$-KMS state at $\beta = 1$.

So, once again, this double role of the modular group $\sigma_t^{\omega_{\text{A PRIORI}}}$ could suggest an interpretation of the Unruh effect in terms of the Noncommutative Bayesian Statistical Inference a statistician performs about the whole noncommutative probability space $(A^{W}_{(M,g_{ab})}, \omega)$ having access only to the local information of A_{H_1}.

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